Problem 1.

(a) This statement is *false*. In order to show that this statement is false, it suffices to show that its negation is true. The negation of this statement reads

$$\forall y \in \mathbb{R}, \exists x \in \mathbb{R}, xy \neq 1.$$

Now let $y \in \mathbb{R}$. Choose $x = 0 \in \mathbb{R}$, then $xy = 0 \neq 1$, as required.

(b) This statement is *true*. Assume that there exists a $q \in \mathbb{Z}$ with $n = 2q + 1$, then $n^2 = (2q+1)^2 = 4q^2 + 4q + 1 = 2(2q^2 + 2q) + 1$, so taking $p = 2q^2 + 2q$ satisfies the desired equality $n^2 = 2p + 1$.

Problem 2.

(a) Since the codomain of g is the domain of f we can talk about $f \circ g$ and $f \circ g : \mathbb{R}^{\geq} \to \mathbb{R}$ is the function given by $(f \circ g)(x) = f(g(x)) = f(x^3 + 1) = (x^3 + 1)^2$. Since the codomain of f is different from the domain of f, $f \circ f$ is undefined.

(b) We claim that f is injective, but not surjective (and consequently also not bijective). To see that f is injective, suppose that $x_1, x_2 \in \mathbb{R}^{\geq}$ satisfy $f(x_1) = f(x_2)$, or equivalently $x_1^2 = x_2^2$. Since $x_1, x_2 \geq 0$ this implies that $x_1 = x_2$, i.e. f is injective. (This is something you probably know from calculus, but we also saw this on the homework, for example in exercise 3.8.(i))

To see that f is not surjective, note that -1 belongs to the codomain $\mathbb{R}$ of f. However, for any $x \in \mathbb{R}^{\geq}$, the quantity $f(x) = x^2$ is the square of a real number and hence nonnegative, so cannot be equal to -1. Therefore, -1 is not in the range of f.

Problem 3. Suppose that $x_1, x_2 \in X$ satisfy $(g \circ f)(x_1) = (g \circ f)(x_2)$, or equivalently $g(f(x_1)) = g(f(x_2))$. Since g is injective we can conclude that $f(x_1) = f(x_2)$. Now, since f is injective we conclude that $x_1 = x_2$, showing that $g \circ f$ is injective.

Problem 4. Divide the 1×1 square into 4 squares of $1/2 \times 1/2$ and number them 1 through 4.:

1	2
3	4

Let S be the set whose elements are the five selected points and define the function $f : S \to \{1, 2, 3, 4\}$ by sending a point $P \in S$ to the number of the square it lies in (if it's on the boundary of two or more such squares, pick any of them). Now, S has 5 elements, whereas $\{1, 2, 3, 4\}$ has only 4, so by the pigeonhole principle f is not injective. Therefore, there exist $P_1, P_2 \in S$ with $P_1 \neq P_2$ and $f(P_1) = f(P_2)$, that is P_1 and P_2 both lie in the same $1/2 \times 1/2$ square. Suppose that the square is placed in the plane with sides parallel to the x- and y-axes. Then each P_i has coordinates (x_i, y_i) where now $|x_1 - x_2| \leq 1/2$ and $|y_1 - y_2| \leq 1/2$. Now, the distance between P_1 and P_2 equals

$$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} = \sqrt{|x_1 - x_2|^2 + |y_1 - y_2|^2} \leq \sqrt{(1/2)^2 + (1/2)^2} = \sqrt{1/2} = \sqrt{2}/2,$$

as desired.

Problem 5. *(As pointed out during the exam, you were asked to find the number of elements in $\{1, 2, \ldots, 100\}$ that are divisible by 3, 5 or both.)* Write $S = \{1, 2, \ldots, 100\}$. Define sets A and B by $A = \{n \in S : 3 \text{ divides } n\}$ and $B = \{n \in S : 5 \text{ divides } n\}$. The question is to find $|A \cup B|$. By the inclusion exclusion principle, $|A \cup B| = |A| + |B| - |A \cap B|$. Firstly, $A = \{3, 6, 9, \ldots, 96, 99\}$, so $|A| = 33$ and $B = \{5, 10, 15, \ldots, 95, 100\}$, so $|B| = 20$. Now, note that

$$A \cap B = \{n \in S : n \in A \text{ and } n \in B\} = \{n \in S : 3 \text{ divides } n \text{ and } 5 \text{ divides } n\} = \{n \in S : 15 \text{ divides } n\}.$$

Therefore, $A \cap B = \{15, 30, 45, 60, 75, 90\}$, so $|A \cap B| = 6$, and the desired answer is $33 + 20 - 6 = 47$.